30/11 — Théorème de Matsushima (Riccardo Perego, GdT Borel)

Thm (Matsushima) G groupe de Lie réel, semi-simple, $[G:G^\circ] < +\infty$, $|Z(G)| < +\infty$, $K \subseteq G$ compact maximal, $\Gamma \subseteq G$ discret, t.q. G/Γ compact. Alors l'application $j^q : (I_G^q)^\Gamma = (\Omega^q(X))^{G^\circ,\Gamma} \longrightarrow H^q(\Gamma)$ est injective $\forall q$, surjective $\forall q \leq m(\mathfrak{g})$, où

$K\backslash G$

$\mathfrak{g} := \mathrm{Lie}(G)$, et $m(\mathfrak{g}) \in \mathbb{N}$ est la constante de Matsushima (voir page 12).

Rmq. $m(\mathfrak{g}) \longrightarrow +\infty$ quand $\dim(\mathfrak{g}) \longrightarrow +\infty$.

Rmq. On peut supposer que $\underbrace{\Gamma_{tors} = \{1\}}_{\text{lemme de Selberg}}$, $G = G^\circ$ et $\underbrace{G \text{ n'a pas de facteurs compactes}}_{\text{on considère des formes sur } X}$.

Si $\Gamma_{tors} = \{1\}$, l'action $X \curvearrowleft \Gamma$ est proprement discontinue $\Rightarrow X/\Gamma$ est une variété lisse.

[**Lemme** $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{k} := \mathrm{Lie}(K)$, $B_{\mathfrak{g}} : \mathrm{Sym}^2(\mathfrak{g}) \to \mathbb{R}$, $B_{\mathfrak{k}} : \mathrm{Sym}^2(\mathfrak{k}) \to \mathbb{R}$ formes de Killing, $\mathfrak{p} := \mathfrak{k}^{\perp}$ par $B_{\mathfrak{g}}$. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Alors on peut choisir des bases :

$$\{\underline{v}_i\}_{i=1}^{m=\dim(\mathfrak{p})} \ (\text{base de } \mathfrak{p}) \quad \text{et} \quad \{\underline{v}_a\}_{a=m+1}^{n=\dim(\mathfrak{g})} \ (\text{base de } \mathfrak{k})$$

t.q. $B_{\mathfrak{g}}(\underline{v}_i, \underline{v}_j) \overset{(I)}{=} \delta_{ij}$, $B_{\mathfrak{g}}(\underline{v}_a, \underline{v}_b) \overset{(II)}{=} -\delta_{ab}$, $B_{\mathfrak{k}}(\underline{v}_a, \underline{v}_b) \overset{(III)}{=} 0$ si $a \neq b$.

Dém $B_{\mathfrak{g}}$ est définie négative sur $\mathfrak{k}$ et positive sur $\mathfrak{p}$, donc Gram-Schmidt donne (I) et (II). Pour (III), $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_N$, $\mathfrak{z} = Z(\mathfrak{k})$, $\mathfrak{k}_j \subseteq \mathfrak{k}$ idéaux simples. Alors si $\underline{v}_a \in \mathfrak{z}$ ou $\underline{v}_b \in \mathfrak{z}$, $B_{\mathfrak{k}}(\underline{v}_a, \underline{v}_b) = \mathrm{tr}_{\mathfrak{k}}([\underline{v}_a, [\underline{v}_b, \cdot]]) = 0$. Aussi si $\underline{v}_a \in \mathfrak{k}_\alpha$, $\underline{v}_b \in \mathfrak{k}_\beta$ pour $\alpha \neq \beta$. Enfin, $\forall \alpha \in \{1, \dots, N\}$, $\exists\, c_\alpha \in \mathbb{R}^{\times}$ t.q. $B_{\mathfrak{k}}(\underline{v}, \underline{w}) = c_\alpha \cdot B_{\mathfrak{g}}(\underline{v}, \underline{w})$, $\forall \underline{v}, \underline{w} \in \mathfrak{k}_\alpha$, donc on a (III).]

Def $\{c^{\nu}_{\lambda\mu}\}^{n}_{\lambda,\mu,\lambda=1}$ définie par $[\underline{v}_\lambda, \underline{v}_\mu] = \sum_{\nu=1}^{n} c^{\nu}_{\lambda\mu} \cdot \underline{v}_\nu$

Def $L(\underline{v},\underline{w}) := B_g(\underline{v},\underline{w}) - B_k(\underline{v},\underline{w})$, $\forall \underline{v},\underline{w} \in k$

Rmq $L(\underline{v},\underline{w}) = tr_p(\underbrace{[\underline{v},[\underline{w},\cdot]]}_{: p \to p})$, $L(\underline{v}_a,\underline{v}_b) = -\sum_{i,j=1}^{m} c^{a}_{ij} c^{b}_{ij}$

Def $R_{ijkl} := -\sum_{a=m+1}^{n} c^{a}_{ij} c^{a}_{kl}$

Def $A(g) := \min\left(-L(\underline{v},\underline{v}) \,\middle|\, \underline{v} \in k, B_g(\underline{v},\underline{v}) = -1\right)$

Rmq. $0 < A(g) \leq 1$

$$
\begin{tikzcd}
G \arrow[r, dashed] \arrow[d] & X \arrow[d] & (\Omega^q_X)^\Gamma \arrow[r, phantom, "\cong"] & (\mathcal{H}^q_X)^\Gamma \\
G/\Gamma \arrow[r, dashed] \arrow[d, no head, bend right] & X/\Gamma & \Omega^q_{X/\Gamma} \arrow[u, "\wr"] \arrow[d, "\wr"] \arrow[r, phantom, "\cong"] & \mathcal{H}^q_{X/\Gamma} \arrow[u] \arrow[d, "\wr"] \\
\Omega^q_{G/\Gamma} \arrow[rr, phantom, "\cong"] & & \Omega^q_0 \arrow[r, phantom, "\cong"] & \mathcal{H}^q_0
\end{tikzcd}
$$

} François

À partir de la base $\{v_\lambda\}_{\lambda=1}^n$ on a des champs de vecteurs $\{V_\lambda\}_{\lambda=1}^n$ sur G et G/Γ.

On a la base duale

$$\underbrace{\{\omega^\lambda\}_{\lambda=1}^n}_{\text{Maurer - Cartan}} \subseteq \Omega^1_{G/\Gamma}.$$

On a une base de $\Omega^q_{G/\Gamma}$, donné par $\{\omega^I\}_{I\in S_q}$, où

$$S_q := \{ I \subseteq \{1,\dots,n\} \mid |I| = q \}$$

$$\omega^I := \omega^{i_1} \wedge \dots \wedge \omega^{i_q},\ I = \{i_1,\dots,i_q\},\ 1 \le i_1 < \dots < i_q \le n.$$

Lemme $\eta \in \Omega^q_{G/H}$, $\eta = \sum_{I \in S_q} \eta_I \cdot \omega^I$, $\eta_I = \eta(V_{i_1}, \dots, V_{i_q}) \in C^\infty(G/H)$. Alors:

(I) $\eta \in \Omega^q_0$ ssi $\mathcal{L}_{V_a}(\eta) = i(V_a)(\eta) = 0$, $\forall a \in \{m+1, \dots, n\}$

(II) $i(V_a)(\eta) = 0$ ssi $\eta_I = 0$, $\forall I \in S_q$ t.q. $a \in I$

(III) $\mathcal{L}_{V_a}(\eta) = 0$ ssi $V_a(\eta_I) = \sum_{v=1}^{q} \sum_{j=1}^{m} (-1)^{v+1} \cdot c^a_{j,i_v} \cdot \eta(V_j, V_{i_1}, \dots, \hat{V}_{i_v}, \dots, V_{i_q})$

(IV) $\eta \in \mathcal{H}^q_0$ ssi $\eta \in \Omega^q_0$ et $\sum_{j=1}^{m} V_j(V_j(\eta_I)) = \sum_{a=m+1}^{n} V_a(V_a(\eta_I))$, $\forall I$

Dém (I) François a fait la même chose pour $G \to X$;

(II) $i(V_a)(\eta)(V_{i_1}, \dots, V_{i_{q-1}}) = \eta(V_a, V_{i_1}, \dots, V_{i_{q-1}}) = \pm \eta_{I'}$, avec $a \in I$.

(III) $\mathcal{L}_{v_a}(\eta) = \sum_I v_a(\eta_I)\, \omega^I + \eta_I \cdot \underbrace{\mathcal{L}_{v_a}(\omega^I)}$

$\mathcal{L}_{v_a}(\omega^I) = \sum_{v=1}^{q} (-1)^{v-1} \cdot (\omega^{i_1} \wedge \dots \wedge \underbrace{\mathcal{L}_{v_a}(\omega^{i_v})} \wedge \dots \wedge \omega^{i_q})$

$\mathcal{L}_{v_a}(\omega^{i_v}) = \cancel{d(\underbrace{i(v_a)(\omega^{i_v})}_{= \omega^{i_v}(v_a) = \delta_{a,i_v}})} + i(v_a)(\underbrace{d(\omega^{i_v})})$

$d(\omega^{i_v}) = -\frac{1}{2} \cdot \sum_{\lambda,\mu=1}^{n} c^{i_v}_{\lambda\mu} \cdot (\omega^\lambda \wedge \omega^\mu)$ (Formule de Maurer-Cartan)

(IV) $\alpha \in \Omega^q_{X|\Gamma}$, $\alpha_0 = \sum_I \alpha_I \cdot \omega^I \Rightarrow (\Delta\alpha)_0 \overset{(+)}{=} -\sum_I C(\alpha_I) \cdot \omega^I$

où $C : C^\infty(G|\Gamma) \to C^\infty(G|\Gamma)$, $C(f) := \sum_{i=1}^{m} v_i(v_i(f)) - \sum_{a=m+1}^{n} v_a(v_a(f))$

Pour montrer (†), il faut $(\Delta(\alpha))_0 = \delta(d\alpha)_0 + d(\delta\alpha)_0$, et décrire les formules explicites pour la pullback des différentielles et codifférentielles données dans Matsushima, (3.1) et Lemma 3.

"On Betti numbers of compact ..."

Démonstration (du théorème) WLOG $\Gamma_{tors} = \{1\}$, $G = G^0$, G sans facteurs compacts,

$$j^q : (I^q_G)^\Gamma \underset{\Gamma \subseteq G = G^0}{=} I^q_G = (\Omega^q_X)^G \underset{\text{François}}{=} (\mathcal{H}^q_X)^G \subseteq \underbrace{(\mathcal{H}^q_X)^\Gamma}_{\begin{subarray}{c}\simeq \mathcal{H}^q_{X/\Gamma} \\ \simeq \mathcal{H}^q_0\end{subarray}} \xrightarrow{\sim} \mathcal{H}^q_{X/\Gamma} \underset{X/\Gamma \text{ compact}}{=} \mathcal{H}^q_{X/\Gamma,(2)} \xrightarrow{\sim} A^q(X/\Gamma) \underset{\text{François}}{=} H^q(\Gamma)$$

j^q est injective $\forall q$.

j^q surjective $\iff \forall \eta \in \mathcal{H}^q_0$, $\eta = \sum_I \eta_I \omega^I$ donc $V_i(\eta_I) = V_a(\eta_I) = 0$, $\forall i = 1, \dots, m$
$a = m+1, \dots, n$.

On a $\sum_i V_i(V_i(\eta_I)) = \sum_a V_a(V_a(\eta_I))$, alors $\{\langle\cdot,\cdot\rangle := (\cdot,\cdot)_{G/\Gamma}\}$

$$\underbrace{\left\langle \sum_{i=1}^{m} V_i(V_i(\eta_I)), \eta_I \right\rangle = \left\langle \sum_{a=m+1}^{n} V_a(V_a(\eta_I)), \eta_I \right\rangle}$$

$$\underbrace{-\sum_{i=1}^{m} \|V_i(\eta_I)\|^2 = -\sum_{a=m+1}^{n} \|V_a(\eta_I)\|^2}$$

$\Rightarrow$ Il suffit d'avoir $V_i(\eta_I) = 0$, $\forall I \in \mathfrak{S}_q$, $i \in \{1, \dots, m\}$.

<u>Idée</u> : Borner la somme $\sum_i \|V_i(\eta_I)\|^2$. On utilise la somme

$\Phi(\eta) := \frac{(q-1)!}{2} \cdot \sum_{i,j=1}^{m} \sum_{I \in \mathfrak{S}_q} \|[V_i, V_j](\eta_I)\|^2$, en deux façons différentes :

$$\Phi(\eta) = \frac{(q-1)!}{2} \cdot \sum_{i,j=1}^{m} \sum_{I \in S_q} \left\langle [v_i, v_j](\eta_I), [v_i, v_j](\eta_I) \right\rangle =$$

$$= \frac{(q-1)!}{2} \cdot \sum_{i,j} \sum_{I} \sum_{a,b=m+1}^{n} c_{ij}^a c_{ij}^b \left\langle v_a(\eta_I), v_b(\eta_I) \right\rangle =$$

$$= \frac{(q-1)!}{2} \cdot \sum_{a,b} \sum_{I} \overline{\left(\sum_{i,j} c_{ij}^a c_{ij}^b \right)} \left\langle v_a(\eta_I), v_b(\eta_I) \right\rangle =$$

$$= \frac{(q-1)!}{2} \cdot \sum_{a,b} \sum_{I} \left(-L(\underline{v_a}, \underline{v_b}) \right) \cdot \left\langle v_a(\eta_I), v_b(\eta_I) \right\rangle =$$

$$= \frac{(q-1)!}{2} \cdot \sum_{a} \sum_{I} \left(-L(\underline{v_a}, \underline{v_a}) \right) \cdot \overline{\left\langle v_a(\eta_I), v_a(\eta_I) \right\rangle} \geq A(g) \cdot \frac{(q-1)!}{2} \cdot \sum_{a} \sum_{I} \| v_a(\eta_I) \|^2 =$$

$$= A(g) \cdot \frac{(q-1)!}{2} \cdot \sum_{i} \sum_{I} \| v_i(\eta_I) \|^2$$

$$U_q := \{1,\dots,n\}^q, \quad \eta_{(J)} := \eta(v_{j_1},\dots,v_{j_q})$$

$$\Phi(q) = \frac{1}{2q}\sum_{i,j=1}^{m}\sum_{J\in U_q}\left\|[v_i,v_j](\eta_{(J)})\right\|^2 =$$

$$= \frac{1}{2q}\sum_{i,j=1}^{m}\sum_{J\in U_q}\sum_{a=m+1}^{n} c_{ij}^a \left\langle v_a(\eta_{(J)}), [v_i,v_j](\eta_{(J)})\right\rangle =$$

$$c_{ij}^a[v_i,v_j] + c_{ji}^a[v_j,v_i] = 2c_{ij}^a(v_iv_j + v_jv_i)$$

$$= \frac{1}{q}\sum_{i,j}\sum_{J}\sum_{a} c_{ij}^a\left\langle \underline{v_a(\eta_{(J)})}, v_i(v_j(\eta_{(J)}))\right\rangle =$$

$$\langle W(\lambda),g\rangle + \langle \lambda, W(g)\rangle = 0$$

$$= \frac{1}{q}\sum_{i,j}\sum_{J}\sum_{k=1}^{m}\sum_{v=1}^{q}(-1)^{v-1}\left(\sum_a c_{ij}^a c_{kj_v}^a\right)\cdot\left\langle \eta_{\underbrace{(k,J)^{\wedge v+1}}}, v_i(v_j(\eta_{(J)}))\right\rangle =$$

$$(k,J)^{\wedge v+1} := (k, j_1,\dots,\hat{j_v},\dots,j_q)$$

$$= \frac{1}{q}\sum_{i,j,k}\sum_{J}\sum_{v}(-1)^{v-1}\left(\sum_a c_{ij}^a c_{kj_v}^a\right)\left\langle v_i(\eta_{(k,J)^{\wedge v+1}}), v_j(\eta_{(J)})\right\rangle = (+)$$

$$(*) = \frac{1}{q} \sum_{i,j,k} \sum_{J} \sum_{u} (-1)^{u-1} R_{i j k_u j_u} \left\langle V_i\left(\eta_{(k_u,J)^{\wedge u+1}}\right), V_j\left(\eta_{(J)}\right)\right\rangle =$$

$$= \frac{1}{q} \sum_{i,j,k} \sum_{J} \sum_{u} R_{i j k j_u} \left\langle V_i\left(\eta_{(k,J)^{\wedge u+1}}\right), V_j\left(\eta_{(j_u,J)^{\wedge u+1}}\right)\right\rangle =$$

$$= \sum_{i,j,k,\ell=1}^{m} \sum_{J' \in U_{q-1}} R_{ijk\ell} \left\langle V_i\left(\eta_{(k,J')}\right), V_j\left(\eta_{(\ell,J')}\right)\right\rangle \geq$$

$$\geq \frac{A(g)\cdot(q-1)!}{2} \sum_{i=1}^{m} \sum_{I \in S_q} \|V_i(\eta_I)\|^2 = \frac{A(g)}{2q} \cdot \sum_{i,j=1}^{m} \sum_{J' \in U_{q-1}} \|V_i(\eta_{(j,J')})\|^2$$

$$0 \geq \sum_{J' \in U_{q-1}} \left(\frac{A(g)}{2q} \cdot \sum_{i,j=1}^{m} \|V_i(\eta_{(j,J')})\|^2 - \sum_{i,j,k,\ell=1}^{m} R_{ijk\ell} \left\langle V_i\left(\eta_{(k,J')}\right), V_j\left(\eta_{(\ell,J')}\right)\right\rangle \right) =$$

$$= \sum_{J' \in U_q} \underbrace{F^q_g}\left(\tilde{\eta}_{(J')}\right), \text{ wo } \tilde{\eta}_{(J')} \overset{(M)}{:=} \sum_{i,j=1}^{m} V_j\left(\eta_{(j,J')}\right) \otimes (\underline{v}_i \otimes \underline{v}_j)$$

$$\in C^\infty(G/\Gamma) \otimes (\mathfrak{p} \otimes \mathfrak{p})$$

$$F^q_g\left(\sum_{ij} f_{ij} \cdot (\underline{v}_i \otimes \underline{v}_j)\right) := \frac{A}{2q} \sum \|f_{ij}\|^2 - \sum_{ijkl} R_{ijkl} \cdot \langle f_{ij}, f_{kl} \rangle$$

$$m(g) := \max\left(\{0\} \cup \{q \mid F^q_g > 0\}\right)$$

$$\Rightarrow \sum_{J' \in U_q} F^q_g\left(\tilde{\eta}_{(J')}\right) < 0 \Rightarrow \tilde{\eta}_{(J')} = 0, \ \forall J' \overset{(M)}{\Rightarrow} V_i(\eta_{(J)}) = 0, \ \forall i = 1,..,m, \ J \in U_q$$

Q.E.D.

Ex. La constante $A(\mathfrak{g})$ est facile à calculer, pour $\mathfrak{g}$ simple. On a $A(\mathfrak{g}) = \dfrac{\dim(\mathfrak{p})}{2\dim(\mathfrak{a})}$.

En particulier, $A(\mathfrak{sl}_n(\mathbb{R})) = \dfrac{n+2}{2n} = \dfrac{1}{2} + \dfrac{1}{n}$ et $A(\mathfrak{sl}_n(\mathbb{C})) = \dfrac{1}{2}$.

Ex. On peut calculer la constante de Matsushima pour $\mathfrak{sl}_n(\mathbb{R})$ et $\mathfrak{sl}_n(\mathbb{C})$, on obtient $m(\mathfrak{sl}_n(\mathbb{R})) = \left\lfloor \dfrac{n+2}{4} \right\rfloor$ et $m(\mathfrak{sl}_n(\mathbb{C})) = \left\lfloor \dfrac{n}{4} \right\rfloor$.

↳ Voir l'exposé de George Boxer.